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壳体中性曲面变形后的度量张量和曲率张量

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摘要：为了从数学角度更好地描述壳体中性曲面如何变形，通过渐近分析和张量分析，给出了当壳体中性曲面发生形变时度量张量和曲率张量改变量的完整表达式．提出的度量张量的改变量与 Ciarlet 给出的一致，而曲率张量的改变量比 Ciarlet 给出的更精确．由于度量张量和曲率张量的改变量是构造 Koiter 型线性、非线性弹性壳体模型的重要组成部分，因此提出新的 Koiter 型非线性弹性壳体模型，理论上比 Ciarlet 的非线性模型误差更小．这为火箭、导弹、航天飞船等国防领域和火车、汽车等工业领域的研究提供了更好的数学基础．

关键词：度量张量；曲率张量；渐近分析；张量分析
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Metric Tensor and Curvature Tensor in Deformed Shell Middle Surface

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Abstract：To mathematically characterize the deformation of the middle surface of a shell，the metric tensor and curvature tensor ought to be investigated in detail. Based on the definitions of metric tensor and curvature tensor，the full expressions of their changes are put forward with asymptotic analysis，where the metric tensor change expression coincides with Ciarlet's，and the curvature tensor change expression is described more accurately than Ciarlet's. Then a new nonlinear shell model of Koiter's type is constructed with smaller error than in Ciarlet's theory for the fields of national defense，such as rockets，missiles，aircrafts，and for the fields of industry，such as trains and cars.

Keywords：metric tensor；curvature tensor；asymptotic analysis；tensor analysis

本文研究了壳体中性面 $\boldsymbol{S}$ 变形为 $\boldsymbol{S}(\boldsymbol{\eta})$ 时，其度量张量和曲面张量是如何变化的．最近，Ciarlet 将度量张量和曲率张量的改变量作为新的未知变量，从微分几何的观念来研究壳体模型[1-3]，因此研究度量张量和曲率张量的改变量显得重要．

20 世纪 60 年代，Koiter 通过渐近分析提出了一个二维的壳体模型，称为 Koiter 模型，有如下形式[4-5]．

求 $\boldsymbol{\eta}\in \boldsymbol{V}_K(\boldsymbol{\omega})=\{\boldsymbol{\eta}=(\eta^i)\in H^1(\boldsymbol{\omega})\times H^1(\boldsymbol{\omega})\times H^2(\boldsymbol{\omega})；\eta^i=\partial_n\eta^3=0 \text{ on } \gamma\}$
满足

$$\frac{\varepsilon}{4}\iint_{\boldsymbol{\omega}} a^{\alpha\beta\sigma\tau}(a_{\sigma\tau}(\boldsymbol{\eta})-a_{\sigma\tau})(a_{\alpha\beta}(\boldsymbol{\zeta})-a_{\alpha\beta})a^{1/2}\mathrm{d}x+\frac{\varepsilon^3}{3}\iint_{\boldsymbol{\omega}} a^{\alpha\beta\sigma\tau}(b_{\sigma\tau}(\boldsymbol{\eta})-b_{\sigma\tau})(b_{\alpha\beta}(\boldsymbol{\zeta})-b_{\alpha\beta})a^{1/2}\mathrm{d}x=\iint_{\boldsymbol{\omega}}\boldsymbol{p}\cdot\boldsymbol{\zeta}a^{1/2}\mathrm{d}x\quad \forall\boldsymbol{\zeta}\in\boldsymbol{V}_K(\boldsymbol{\omega})$$

式中：$a^{\alpha\beta\sigma\tau}$ 是二维弹性张量的逆变分支；$a_{\alpha\beta}(\boldsymbol{\zeta})-a_{\alpha\beta}$ 和 $b_{\alpha\beta}(\boldsymbol{\zeta})-b_{\alpha\beta}$ 分别是与壳体中性面 $\boldsymbol{S}$ 上的位移场 $\boldsymbol{\eta}$ 相关的度量张量和曲率张量改变量的协变分支；$\boldsymbol{p}\in \boldsymbol{L}^2(\boldsymbol{\omega})$ 是外力．显而易见，Koiter 模型的线性、非线性依赖于 $a_{\alpha\beta}(\boldsymbol{\zeta})-a_{\alpha\beta}$ 和 $b_{\alpha\beta}(\boldsymbol{\zeta})-b_{\alpha\beta}$ 的线性、非线性．Ciarlet 给出了 $a_{\alpha\beta}(\boldsymbol{\zeta})-a_{\alpha\beta}$ 和 $b_{\alpha\beta}(\boldsymbol{\zeta})-$

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$b_{\alpha\beta}$的逼近表达式,从而提出了 Koiter 型非线性弹性壳体模型[6-7].

本文给出了度量张量和曲率张量改变量的精确表达式,前者与 Ciarlet 给出的一致,后者比 Ciarlet 给出的更精确. 因此,我们也可以构造出新的 Koiter 型非线性弹性壳体模型,从理论上讲,新的模型比 Ciarlet的模型更精确,误差更小,这将在另一篇文章中讨论.

1 预备知识

文中所有拉丁指标 $i,j,k\cdots$在集合$\{1,2,3\}$中取值,希腊指标 $\alpha,\beta,\gamma\cdots$在$\{1,2\}$中取值,此外 Einstain 指标缩并法则也在文中使用. $\boldsymbol{a},\boldsymbol{b}\in R^3$,其内积和外积分别记为 $\boldsymbol{a}\cdot\boldsymbol{b}$ 和 $\boldsymbol{a}\wedge\boldsymbol{b}$. 令 $\boldsymbol{\omega}$ 是R^2 中的一个有界连通开子集,其边界 $\boldsymbol{\gamma}$ 是 Lipschitz 连续,且 $\boldsymbol{\omega}$ 在 $\boldsymbol{\gamma}$ 一侧. $\boldsymbol{y}=(y^\alpha)$是 $\bar{\boldsymbol{\omega}}$ 的一个内点,$\partial_\alpha:=\partial/\partial y^\alpha$. 映射 $\boldsymbol{\theta}\in C^2(\bar{\boldsymbol{\omega}};R^3)$使得向量 $\boldsymbol{e}_\alpha(\boldsymbol{y}):=\partial_\alpha\boldsymbol{\theta}(\boldsymbol{y})$在点 $\boldsymbol{y}\in\bar{\boldsymbol{\omega}}$ 处线性独立. 这两个向量在点 $\boldsymbol{\theta}(\boldsymbol{y})$处张成曲面 $\boldsymbol{S}:=\boldsymbol{\theta}(\bar{\boldsymbol{\omega}})$的切平面,曲面 $\boldsymbol{S}$ 在点 $\boldsymbol{\theta}(\boldsymbol{y})$的单位法向为 $\boldsymbol{n}(\boldsymbol{y})=\boldsymbol{e}_3(\boldsymbol{y}):=\dfrac{\boldsymbol{e}_1(\boldsymbol{y})\wedge\boldsymbol{e}_2(\boldsymbol{y})}{|\boldsymbol{e}_1(\boldsymbol{y})\wedge\boldsymbol{e}_2(\boldsymbol{y})|}$,则向量 $\boldsymbol{e}_i(\boldsymbol{y})$构成点 $\boldsymbol{\theta}(\boldsymbol{y})$处的协变基,由关系式 $\boldsymbol{e}^i(\boldsymbol{y})\cdot\boldsymbol{e}_j(\boldsymbol{y})=\delta^i_j$ 确定的$\boldsymbol{e}^i(\boldsymbol{y})$也构成点 $\boldsymbol{\theta}(\boldsymbol{y})\in\boldsymbol{S}$ 处的逆变基,其中 δ^i_j 是 Kronecker 符号(注意 $\boldsymbol{e}^3(\boldsymbol{y})=\boldsymbol{e}_3(\boldsymbol{y})$和 $\boldsymbol{e}^\alpha(\boldsymbol{y})$也构成 $\boldsymbol{S}$ 在点 $\boldsymbol{\theta}(\boldsymbol{y})$处的且平面). $\boldsymbol{S}$ 上的度量张量的协变、逆变分支 $a_{\alpha\beta}$和 $a^{\alpha\beta}$,Christoffel 记号为$\overset{*}{\Gamma}{}^\sigma_{\alpha\beta}$,曲率张量的协变、混合分支 $b_{\alpha\beta}$和 b^β_α 定义如下(显然它们依赖于变量 $\boldsymbol{y}\in\bar{\boldsymbol{\omega}}$,此处 $\boldsymbol{y}\in\bar{\boldsymbol{\omega}}$ 被省掉)

$$a_{\alpha\beta}=\boldsymbol{e}_\alpha\cdot\boldsymbol{e}_\beta;\ a^{\alpha\beta}=\boldsymbol{e}^\alpha\cdot\boldsymbol{e}^\beta;\ a^{\alpha\beta}a_{\beta\lambda}=\delta^\alpha_\lambda$$

$$\overset{*}{\Gamma}{}^\sigma_{\alpha\beta}=\boldsymbol{e}^\sigma\cdot\partial_\beta\boldsymbol{e}_\alpha;\ b_{\alpha\beta}=n\cdot\partial_\beta\boldsymbol{e}_\alpha;\ b^\beta_\alpha=a^{\beta\sigma}b_{\sigma\alpha}$$

度量张量的行列式为 $\boldsymbol{a}:=\det(a_{\alpha\beta})$,此外给出置换张量如下[8]

$$\varepsilon_{\alpha\beta}=\begin{cases}a^{\frac{1}{2}}\\-a^{\frac{1}{2}}\\0\end{cases}$$

$$\varepsilon^{\alpha\beta}=\begin{cases}a^{-\frac{1}{2}} & (\alpha,\beta):(1,2)\ \text{的偶置换}\\-a^{-\frac{1}{2}} & (\alpha,\beta):(1,2)\ \text{的奇置换}\\0 & \text{其他}\end{cases}$$

$\boldsymbol{S}$ 上的协变导数定义[8]为 $\overset{*}{\nabla}_\alpha b^\sigma_\tau=\partial b^\sigma_\tau/\partial y^\alpha+\overset{*}{\Gamma}{}^\sigma_{\alpha\beta}b^\beta_\tau-\overset{*}{\Gamma}{}^\beta_{\alpha\tau}b^\sigma_\beta$.

2 主要结论

给出壳体中性面发生形变时度量张量改变量的完整表达式.

定理 1 假定 $\boldsymbol{\eta}=\eta^i\boldsymbol{e}_i$ 是定义在 $\boldsymbol{S}=\boldsymbol{\theta}(\bar{\boldsymbol{\omega}})$上的向量场,$a_{\alpha\beta}$和 $a_{\alpha\beta}(\boldsymbol{\eta})$分别是定义在中性面 $\boldsymbol{S}=\boldsymbol{\theta}(\bar{\boldsymbol{\omega}})$和 $\boldsymbol{S}(\boldsymbol{\eta})=(\boldsymbol{\theta}+\boldsymbol{\eta})(\bar{\boldsymbol{\omega}})$上的度量张量,则有如下式子成立

$$a_{\alpha\beta}(\boldsymbol{\eta})-a_{\alpha\beta}=2\gamma_{\alpha\beta}(\boldsymbol{\eta})+\varphi_{\alpha\beta}(\boldsymbol{\eta})$$

其中

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}):=\frac{1}{2}(a_{\beta\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma+a_{\alpha\lambda}\overset{0}{\nabla}_\beta\eta^\lambda)$$

$$\varphi_{\alpha\beta}(\boldsymbol{\eta}):=a_{ij}\overset{0}{\nabla}_\alpha\eta^i\overset{0}{\nabla}_\beta\eta^j$$

$$a_{ij}=\{a_{\alpha\beta},i=\alpha,j=\beta,a_{3\alpha}=a_{\alpha3}=0,a_{33}=0\}$$

$$\overset{0}{\nabla}_\alpha\eta^\sigma=\overset{*}{\nabla}_\alpha\eta^\sigma-b^\sigma_\alpha\eta^3;\overset{0}{\nabla}_\alpha\eta^3=\overset{*}{\nabla}_\alpha\eta^3+b_{\alpha\lambda}\eta^\lambda$$

如果 $\boldsymbol{\eta}$ 充分小,则 $\varphi_{\alpha\beta}(\boldsymbol{\eta})$是高阶项.

证明 令 $\boldsymbol{\eta}$ 是$\boldsymbol{S}$ 上$\boldsymbol{S}$-坐标系的基向量,即 $\boldsymbol{\eta}=\eta^\lambda\boldsymbol{e}_\lambda+\eta^3\boldsymbol{n}$,则

$$\eta_\alpha=\partial_\alpha\boldsymbol{\eta}=\partial_\alpha(\eta^\lambda\boldsymbol{e}_\lambda+\eta^3\boldsymbol{n})=$$
$$\partial_\alpha\eta^\lambda\boldsymbol{e}_\lambda+\eta^\lambda\boldsymbol{e}_{\alpha\lambda}+\partial_\alpha\eta^3\boldsymbol{n}+\eta^3n_\alpha$$

由 Gauss 和 Weingarten 公式,有

$$e_{\alpha\lambda}=\overset{*}{\Gamma}{}^\sigma_{\alpha\lambda}e_\sigma+b_{\alpha\lambda}\boldsymbol{n},\boldsymbol{n}_\alpha=-b^\sigma_\alpha\boldsymbol{e}_\sigma$$

则 $\eta_\alpha=(\overset{*}{\nabla}_\alpha\eta^\sigma-b^\sigma_\alpha\eta^3)\boldsymbol{e}_\sigma+(\overset{*}{\nabla}_\alpha\eta^3+b_{\alpha\lambda}\eta^\lambda)\boldsymbol{n}$. 令 $\overset{0}{\nabla}_\alpha\eta^\sigma=\overset{*}{\nabla}_\alpha\eta^\sigma-b^\beta_\alpha\eta^3$,$\overset{0}{\nabla}_\alpha\eta^3=\overset{*}{\nabla}_\alpha\eta^3+b_{\alpha\lambda}\eta^\lambda$,则 η_α 变为

$$\eta_\alpha=\overset{0}{\nabla}_\alpha\eta^\sigma e_\sigma+\overset{0}{\nabla}_\alpha\eta^3n \tag{1}$$

因此,变形中性面 $\boldsymbol{S}(\boldsymbol{\eta})$上的基向量是

$$\boldsymbol{e}_\alpha(\boldsymbol{\eta})=\partial_\alpha(\boldsymbol{\theta}+\boldsymbol{\eta})=\boldsymbol{e}_\alpha+\overset{0}{\nabla}_\alpha\eta^\sigma\boldsymbol{e}_\sigma+\overset{0}{\nabla}_\alpha\eta^3\boldsymbol{n}=$$
$$(\delta^\sigma_\alpha+\overset{0}{\nabla}_\alpha\eta^\sigma)\boldsymbol{e}_\sigma+\overset{0}{\nabla}_\alpha\eta^3\boldsymbol{n} \tag{2}$$

则 $\boldsymbol{S}(\boldsymbol{\eta})$上的度量张量为

$$a_{\alpha\beta}(\eta)=a_{\alpha\beta}+a_{\alpha\lambda}\overset{0}{\nabla}_\beta\eta^\lambda+a_{\beta\sigma}\overset{0}{\nabla}_\beta\eta^\lambda+$$
$$a_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^\lambda+\overset{0}{\nabla}_\alpha\eta^3\overset{0}{\nabla}_\beta\eta^3 \tag{3}$$

由 $\overset{0}{\nabla}_{\alpha\beta\gamma}=0$,有

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}):=\frac{1}{2}(\overset{0}{\nabla}_\alpha\eta_\beta+\overset{0}{\nabla}_\beta\eta_\alpha)=$$
$$\frac{1}{2}(a_{\beta\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma+a_{\alpha\lambda}\overset{0}{\nabla}_\beta\eta^\lambda)$$

令

$$\varphi_{\alpha\beta}(\boldsymbol{\eta}):=a_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^\lambda+\overset{0}{\nabla}_\alpha\eta^3\overset{0}{\nabla}_\beta\eta^3=a_{ij}\eta^i\overset{0}{\nabla}_\beta\eta^j$$

则式(3)改写为 $a_{\alpha\beta}(\boldsymbol{\eta})-a_{\alpha\beta}=2\gamma_{\alpha\beta}(\boldsymbol{\eta})+\varphi_{\alpha\beta}(\boldsymbol{\eta})$. 证毕.

定理 1 说明度量张量的改变量由两部分组成,一部分是 $\boldsymbol{S}$ 上的变形张量 $\gamma_{\alpha\beta}(\boldsymbol{\eta})$,另一部分是关于 $\boldsymbol{\eta}$ 的二阶多项式 $\varphi_{\alpha\beta}(\boldsymbol{\eta})=O(|\boldsymbol{\eta}|^2)$.

为了得到曲率张量的改变量,先给出两个引理.

第 1 个引理给出壳体中性面发生形变时度量张量行列式的变化情况，第 2 个引理研究壳体中性面发生形变后的单位法向量.

引理 1　在定理 1 的假设下，$\boldsymbol{a}$ 和 $\boldsymbol{a}(\boldsymbol{\eta})$ 分别是 $\boldsymbol{S}=\boldsymbol{\theta}(\bar{\boldsymbol{\omega}})$ 和 $\boldsymbol{S}(\boldsymbol{\eta})=(\boldsymbol{\theta}+\boldsymbol{\eta})(\bar{\boldsymbol{\omega}})$ 上度量张量的行列式. 令 $q^{-2}:=\boldsymbol{a}(\boldsymbol{\eta})/\boldsymbol{a}$，则有 $q^{-2}:=1+2\gamma_0(\boldsymbol{\eta})+\pi(\boldsymbol{\eta})$ 成立，其中 $\gamma_0(\boldsymbol{\eta})$ 和 $\pi(\boldsymbol{\eta})$ 定义如下

$$\left.\begin{array}{c}\gamma_0(\boldsymbol{\eta})=a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta});\ \varphi_0(\boldsymbol{\eta})=a^{\alpha\beta}\varphi_{\alpha\beta}(\boldsymbol{\eta})\\ \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\lambda\sigma}(\boldsymbol{\eta})=\dfrac{1}{a}\det(\gamma_{\alpha\beta}(\boldsymbol{\eta}))\\ \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\varphi_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})=\dfrac{1}{a}\det(\varphi_{\alpha\beta}(\boldsymbol{\eta}))\\ \pi(\boldsymbol{\eta})=\varphi_0(\boldsymbol{\eta})+2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})+\\ \dfrac{2}{a}\det(\gamma_{\alpha\beta}(\boldsymbol{\eta}))+\dfrac{1}{2a}\det(\varphi_{\alpha\beta}(\boldsymbol{\eta}))=o(|\boldsymbol{\eta}|^2)\end{array}\right\}\tag{4}$$

证明　由定理 1，$\boldsymbol{S}(\boldsymbol{\eta})=(\boldsymbol{\theta}+\boldsymbol{\eta})(\bar{\boldsymbol{\omega}})$ 上的度量张量的行列式为

$$\begin{aligned}a(\boldsymbol{\eta})=\;&a+a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma(\boldsymbol{\eta})}(a_{\alpha\beta}\gamma_{\lambda\sigma}(\boldsymbol{\eta})+a_{\lambda\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta}))+\\&\frac{1}{2}a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}(a_{\alpha\beta}\varphi_{\lambda\sigma}(\boldsymbol{\eta})+a_{\lambda\sigma}\varphi_{\alpha\beta}(\boldsymbol{\eta}))+\\&a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}(\gamma_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})+\gamma_{\lambda\sigma}(\boldsymbol{\eta})\varphi_{\alpha\beta}(\boldsymbol{\eta}))+\\&2a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\lambda\sigma}(\boldsymbol{\eta})+\frac{1}{2}a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\varphi_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})\end{aligned}\tag{5}$$

令 $\gamma_0(\boldsymbol{\eta}):=a^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})$，$\varphi_0(\boldsymbol{\eta}):=a^{\alpha\beta}\varphi_{\alpha\beta}(\boldsymbol{\eta})$. 由置换张量的反对称性 $\varepsilon^{\alpha\beta}=-\varepsilon^{\beta\alpha}$ 和 $\gamma_{\alpha\beta}$，$\varphi_{\alpha\beta}$ 的对称性 $\gamma_{\alpha\beta}=\gamma_{\beta\alpha}$，$\varphi_{\alpha\beta}=\varphi_{\beta\alpha}$，则式(5)变为

$$\begin{aligned}a(\boldsymbol{\eta})=\;&a+\\&2a\gamma_0(\boldsymbol{\eta})+a\varphi_0(\boldsymbol{\eta})+2a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})+\\&a\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\left(2\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\lambda\sigma}(\boldsymbol{\eta})+\frac{1}{2}\varphi_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})\right)\end{aligned}$$

因此

$$\begin{aligned}q^{-2}:=\;&\frac{a(\boldsymbol{\eta})}{a}=\\&1+2\gamma_0(\boldsymbol{\eta})+\varphi_0(\boldsymbol{\eta})+2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})+\\&\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\left(2\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\lambda\sigma}(\boldsymbol{\eta})+\frac{1}{2}\varphi_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})\right)\end{aligned}$$

由

$$\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\alpha\beta}(\boldsymbol{\eta})\gamma_{\lambda\sigma}(\boldsymbol{\eta})=\frac{1}{a}\det(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$$

$$\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\varphi_{\alpha\beta}(\boldsymbol{\eta})\varphi_{\lambda\sigma}(\boldsymbol{\eta})=\frac{1}{a}\det(\varphi_{\alpha\beta}(\boldsymbol{\eta}))$$

最终 $q^{-2}:=1+2\gamma_0(\boldsymbol{\eta})+\pi(\boldsymbol{\eta})$，其中 $\gamma_0(\boldsymbol{\eta})$ 和 $\pi(\boldsymbol{\eta})$ 由式(4)给出. 证毕.

引理 2　在定理 1 的假设下，n 和 $n(\boldsymbol{\eta})$ 分别是 $\boldsymbol{S}=\boldsymbol{\theta}(\bar{\boldsymbol{\omega}})$ 和 $\boldsymbol{S}(\boldsymbol{\eta})=(\boldsymbol{\theta}+\boldsymbol{\eta})(\bar{\boldsymbol{\omega}})$ 上的外法向，则有如下公式成立

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{\eta})=\;&q[(1+\gamma_0(\boldsymbol{\eta})+\det(\overset{0}{\nabla}_\alpha\eta^\beta))\boldsymbol{n}+\\&(-\delta^\alpha_\nu+\varepsilon^{\alpha\beta}\varepsilon_{\sigma\nu}\overset{0}{\nabla}_\alpha\eta^\sigma)\overset{0}{\nabla}_\alpha\eta^3\boldsymbol{e}^\nu]\end{aligned}$$

证明　由

$$\boldsymbol{n}(\boldsymbol{\eta})=\frac{1}{2}\varepsilon^{\alpha\beta}(\boldsymbol{\eta})\boldsymbol{e}_\alpha(\boldsymbol{\eta})\wedge\boldsymbol{e}_\beta(\boldsymbol{\eta})\tag{6}$$

将式(2)带入式(6)，应用 $\varepsilon^{\alpha\beta}(\boldsymbol{\eta})=\left(\dfrac{a}{a(\boldsymbol{\eta})}\right)^{1/2}\varepsilon^{\alpha\beta}$，有

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{\eta})=\;&\frac{1}{2}q\varepsilon^{\alpha\beta}[(\delta^\lambda_\alpha+\overset{0}{\nabla}_\alpha\eta^\lambda)(\delta^\sigma_\beta+\overset{0}{\nabla}_\beta\eta^\sigma)\boldsymbol{e}_\lambda\wedge\boldsymbol{e}_\sigma+\\&(\delta^\lambda_\alpha+\overset{0}{\nabla}_\alpha\eta^\lambda)\overset{0}{\nabla}_\beta\eta^3\boldsymbol{e}_\lambda\wedge\boldsymbol{n}+\\&(\delta^\sigma_\beta+\overset{0}{\nabla}_\beta\eta^\sigma)\overset{0}{\nabla}_\alpha\eta^3\boldsymbol{n}\wedge e_\sigma\end{aligned}$$

由 $\boldsymbol{e}_\alpha\wedge\boldsymbol{e}_\beta=\varepsilon_{\alpha\beta}\boldsymbol{n}$，$\boldsymbol{n}\wedge\boldsymbol{e}_\alpha=\varepsilon_{\alpha\beta}\boldsymbol{e}^\beta$，$\varepsilon^{\alpha\beta}\varepsilon_{\alpha\lambda}=\delta^\beta_\lambda$，则

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{\eta})=\;&\frac{1}{2}q\{[2+2\overset{0}{\nabla}_\beta\eta^\beta+\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\lambda\overset{0}{\nabla}_\beta\eta^\sigma]\boldsymbol{n}+\\&[(-\delta^\alpha_\nu+\varepsilon^{\alpha\beta}\varepsilon_{\sigma\nu}\overset{0}{\nabla}_\beta\eta^\sigma)\overset{0}{\nabla}_\alpha\eta^3-\\&(\delta^\beta_\nu+\varepsilon^{\alpha\beta}\varepsilon_{\lambda\nu}\overset{0}{\nabla}_\alpha\eta^\lambda)\overset{0}{\nabla}_\beta\eta^3]\boldsymbol{e}^\nu\}\end{aligned}$$

由

$$\begin{aligned}\varepsilon^{\alpha\beta}\overset{0}{\nabla}_\beta\eta^\lambda\overset{0}{\nabla}_\alpha\eta^3=\;&-\varepsilon^{\beta\alpha}\overset{0}{\nabla}_\beta\eta^\lambda\overset{0}{\nabla}_\alpha\eta^3=\\&-\varepsilon^{\alpha\beta}\overset{0}{\nabla}_\alpha\eta^\lambda\overset{0}{\nabla}_\beta\eta^3\end{aligned}\tag{7}$$

则 $\boldsymbol{n}(\boldsymbol{\eta})$ 变为

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{\eta})=\;&q\Big[\Big(1+\overset{0}{\nabla}_\beta\eta^\beta+\frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\lambda\overset{0}{\nabla}_\beta\eta^\sigma\Big)\boldsymbol{n}+\\&(-\overset{0}{\nabla}_\nu\eta^3+\varepsilon^{\alpha\beta}\varepsilon_{\sigma\nu}\overset{0}{\nabla}_\beta\eta^\sigma\overset{0}{\nabla}_\alpha\eta^3)\boldsymbol{e}^\nu\Big]\end{aligned}$$

注意 $\overset{0}{\nabla}_\beta\eta^\beta=\overset{*}{\nabla}_\beta\eta^\beta-2H\eta^3=\gamma_0(\eta)$ 和 $\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\lambda\overset{0}{\nabla}_\beta\eta^\sigma=2\det(\overset{0}{\nabla}_\alpha\eta^\beta)$，因此

$$\begin{aligned}\boldsymbol{n}(\boldsymbol{\eta})=\;&q[(1+\gamma_0(\boldsymbol{\eta})+\det(\overset{0}{\nabla}_\alpha\eta^\beta))\boldsymbol{n}+\\&(-\delta^\alpha_\nu+\varepsilon^{\alpha\beta}\varepsilon_{\sigma\nu}\overset{0}{\nabla}_\beta\eta^\sigma)\overset{0}{\nabla}_\alpha\eta^3\boldsymbol{e}^\nu]\end{aligned}\tag{8}$$

定理 2　在定理 1 的假设下，令 $b_{\alpha\beta}$ 和 $b_{\alpha\beta}(\boldsymbol{\eta})$ 分别是 $\boldsymbol{S}=\boldsymbol{\theta}(\bar{\boldsymbol{\omega}})$ 和 $\boldsymbol{S}(\boldsymbol{\eta})=(\boldsymbol{\theta}+\boldsymbol{\eta})(\bar{\boldsymbol{\omega}})$ 上的曲率张量，则有 $b_{\alpha\beta}(\boldsymbol{\eta})-b_{\alpha\beta}=\rho_{\alpha\beta}(\boldsymbol{\eta})+\Psi_{\alpha\beta}(\boldsymbol{\eta})$，其中 $\rho_{\alpha\beta}(\boldsymbol{\eta})$ 和 $\Psi_{\alpha\beta}(\boldsymbol{\eta})$ 定义如下

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}):=\;&\frac{1}{2}[\overset{*}{\nabla}_\beta\overset{0}{\nabla}_\alpha\eta^3+\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^3+\\&b_{\beta\lambda}\overset{0}{\nabla}_\alpha\eta^\lambda+b_{\alpha\lambda}\overset{0}{\nabla}_\beta\eta^\lambda]\end{aligned}$$

$$\begin{aligned}\Psi_{\alpha\beta}(\boldsymbol{\eta}):=\;&(q-1)(b_{\alpha\beta}+\rho_{\alpha\beta}(\boldsymbol{\eta}))+\\&q[b_{\alpha\beta}\gamma_0(\boldsymbol{\eta})+N^2_{\alpha\beta}(\boldsymbol{\eta})+N^3_{\alpha\beta}(\boldsymbol{\eta})]\end{aligned}$$

$$\begin{aligned}N^2_{\alpha\beta}(\boldsymbol{\eta}):=\;&b_{\alpha\beta}\det(\overset{0}{\nabla}_\alpha\eta^\beta)-\Phi^\sigma_{\alpha\beta}(\eta)\overset{0}{\nabla}_\sigma\eta^3+\\&\gamma_0(\eta)(\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^\sigma_{\alpha\beta}\overset{0}{\nabla}_\sigma\eta^3)+\end{aligned}$$

$$(\varepsilon^{\nu\mu}\varepsilon_{\sigma\lambda}-\delta_{\sigma}^{\mu}\delta_{\lambda}^{\nu})\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\nu}\eta^{\sigma}\overset{0}{\nabla}_{\mu}\eta^{\sigma}$$

$$N_{\alpha\beta}^{3}(\boldsymbol{\eta}):=(\rho_{\alpha\beta}(\boldsymbol{\eta})+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\lambda}\eta^{3})\det(\overset{0}{\nabla}_{\alpha}\eta^{\beta})+$$
$$(\Phi_{\alpha\beta}^{\sigma}(\boldsymbol{\eta})+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\lambda}\eta^{\sigma})\varepsilon^{\nu\mu}\varepsilon_{\sigma\tau}\overset{0}{\nabla}_{\nu}\eta^{\tau}\overset{0}{\nabla}_{\mu}\eta^{3})$$

$$\Phi_{\alpha\beta}^{\sigma}(\boldsymbol{\eta}):=$$
$$\frac{1}{2}[\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\sigma}+\overset{*}{\nabla}_{\alpha}\overset{0}{\nabla}_{\beta}\eta^{\sigma}+b_{\beta}^{\sigma}\overset{0}{\nabla}_{\alpha}\eta^{3}+b_{\alpha}^{\sigma}\overset{0}{\nabla}_{\beta}\eta^{3}]$$

备注:如果 $\boldsymbol{\eta}$ 充分小,则 $\varphi_{\alpha\beta}(\boldsymbol{\eta})$是高阶项.

证明 令 $e_{\alpha\beta}=\partial_{\alpha}\boldsymbol{e}_{\beta}$,由 Gauss 和 Weingarten 公式,有 $e_{\alpha\beta}=\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\boldsymbol{e}_{\lambda}+b_{\alpha\beta}\boldsymbol{n}$. 中性面变形后,$e_{\alpha\beta}$变为

$$e_{\alpha\beta}(\boldsymbol{\eta})=\alpha_{\beta}(\boldsymbol{e}_{\alpha}+\boldsymbol{\eta}_{\alpha})=$$
$$e_{\alpha\beta}+\partial_{\beta}\eta_{\alpha}=\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\boldsymbol{e}_{\lambda}+b_{\alpha\beta}\boldsymbol{n}+\partial_{\beta}\eta_{\alpha} \tag{9}$$

由式(1),有

$$\partial_{\beta}\eta_{\alpha}=[\overset{0}{\nabla}_{\alpha}\eta^{\sigma}\overset{*}{\Gamma}{}_{\sigma\beta}^{\lambda}+\partial_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}-\overset{0}{\nabla}_{\alpha}\eta^{3}b_{\beta}^{\lambda}]\boldsymbol{e}_{\lambda}+$$
$$[\overset{0}{\nabla}_{\alpha}\eta^{\sigma}b_{\sigma\beta}+\partial_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{3}]\boldsymbol{n}$$

由

$$\partial_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}=\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}-\overset{*}{\Gamma}{}_{\sigma\beta}^{\lambda}\overset{0}{\nabla}_{\alpha}\eta^{\sigma}+\overset{*}{\Gamma}{}_{\sigma\beta}^{\sigma}\overset{0}{\nabla}_{\sigma}\eta^{\lambda}$$

且$\partial_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{3}=\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{3}+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\lambda}\eta^{3}$,则有

$$\partial_{\beta}\eta_{\alpha}=(\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\sigma}\eta^{\lambda}-\overset{0}{\nabla}_{\alpha}\eta^{3}b_{\beta}^{\lambda})\boldsymbol{e}_{\lambda}+$$
$$(\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{3}+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\lambda}\eta^{3}+\overset{0}{\nabla}_{\alpha}\eta^{\sigma}b_{\sigma\beta})\boldsymbol{n} \tag{10}$$

将式(10)带入式(9),则有

$$e_{\alpha\beta}(\boldsymbol{\eta})=(\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\sigma}\eta^{\lambda}+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}-\overset{0}{\nabla}_{\alpha}\eta^{3}b_{\beta}^{\lambda})\boldsymbol{e}_{\lambda}+$$
$$(\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{3}+\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}\overset{0}{\nabla}_{\lambda}\eta^{3}+\overset{0}{\nabla}_{\alpha}\eta^{\sigma}b_{\sigma\beta}+b_{\alpha\beta})\boldsymbol{n}$$

类似地可以得到 $e_{\beta\alpha}(\boldsymbol{\eta})$,因此

$$e_{\alpha\beta}(\boldsymbol{\eta})=\frac{1}{2}(e_{\alpha\beta}(\boldsymbol{\eta})+e_{\beta\alpha}(\boldsymbol{\eta}))$$

令

$$\rho_{\alpha\beta}(\boldsymbol{\eta}):=$$
$$\frac{1}{2}[(\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{3}+\overset{*}{\nabla}_{\alpha}\overset{0}{\nabla}_{\beta}\eta^{3}+b_{\beta\lambda}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}+b_{\alpha\lambda}\overset{0}{\nabla}_{\beta}\eta^{\lambda}]$$

$$\Phi_{\alpha\beta}^{\sigma}(\boldsymbol{\eta}):=$$
$$\frac{1}{2}[(\overset{*}{\nabla}_{\beta}\overset{0}{\nabla}_{\alpha}\eta^{\sigma}+\overset{*}{\nabla}_{\alpha}\overset{0}{\nabla}_{\beta}\eta^{\sigma}+b_{\beta}^{\sigma}\overset{0}{\nabla}_{\alpha}\eta^{3}+b_{\alpha}^{\sigma}\overset{0}{\nabla}_{\beta}\eta^{3}]$$

则

$$e_{\alpha\beta}(\boldsymbol{\eta})=(\overset{*}{\Gamma}{}_{\alpha\beta}^{\lambda}(\delta_{\lambda}^{\sigma}+\overset{0}{\nabla}_{\lambda}\eta^{\sigma})+\Phi_{\alpha\beta}^{\sigma}(\boldsymbol{\eta}))\boldsymbol{e}_{\sigma}+$$
$$(b_{\alpha\beta}+\rho_{\alpha\beta}(\boldsymbol{\eta})+\overset{*}{\Gamma}{}_{\alpha\beta}^{\sigma}\overset{0}{\nabla}_{\sigma}\eta^{3})\boldsymbol{n} \tag{11}$$

由于 $b_{\alpha\beta}(\boldsymbol{\eta})=e_{\alpha\beta}(\boldsymbol{\eta})\cdot\boldsymbol{n}(\boldsymbol{\eta})$,由式(7)和式(11)有$b_{\alpha\beta}(\boldsymbol{\eta})=b_{\alpha\beta}+\rho_{\alpha\beta}(\boldsymbol{\eta})+\Psi_{\alpha\beta}(\eta)$,其中 $\rho_{\alpha\beta}(\boldsymbol{\eta})$ 和 $\Psi_{\alpha\beta}(\boldsymbol{\eta})$由式(8)定义. 证毕.

定理 2 表明,曲率张量的改变量由两部分组成,一部分是线性化部分 $\rho_{\alpha\beta}(\boldsymbol{\eta})$,另一部分是关于 η 的三阶多项式 $\Psi_{\alpha\beta}(\boldsymbol{\eta})=O(|\boldsymbol{\eta}|^{3})$.

3 结 论

由 $a_{\alpha\beta}$的定义,首先给出了壳体中性面发生形变时的度量张量改变量的完整形式. 这个结果与 Ciarlet 的表达式一致. 接着,为了给出曲率张量改变量的完整形式,证明了两个引理. 最后,由两个引理和 $b_{\alpha\beta}$的定义,给出了曲率张量改变量的完整形式,这比 Ciarlet 的表达式更精确、更完整. 因为 Ciarlet 是用$R_{\alpha\beta}^{\sharp}$代替$R_{\alpha\beta}$,给出的是曲率张量改变量的逼近形式.

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